Dynamical Numerics for Numerical Dynamics

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continuous dynamical system:

$$\dot{u}(t) = f(u(t)), \quad u(0) = u_0, \quad n \in \mathbb{R}^+, \mathbb{R}, u \in \mathbb{R}^n \qquad (1)$$

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- In most modeling situations, however, f is nonlinear and in general analytic solutions are not available.

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Both approaches have limitations...

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So, not only do we have the issue of floating point error, we also have the issue of whether interpolating the discretized system produces behaviour sufficiently close to the behaviour of the model

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- which contains the logistic map.

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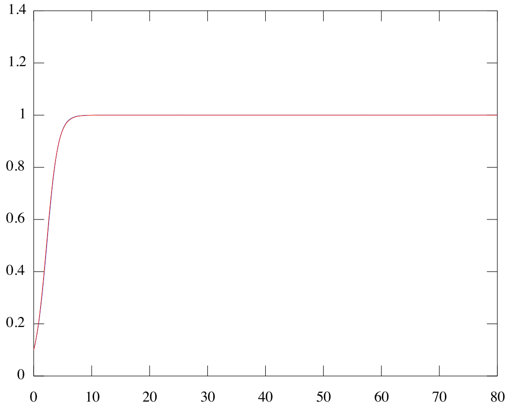
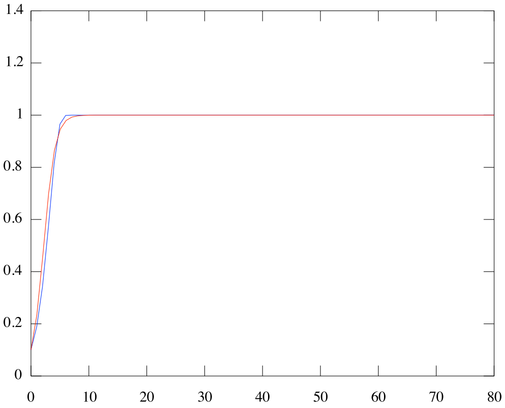
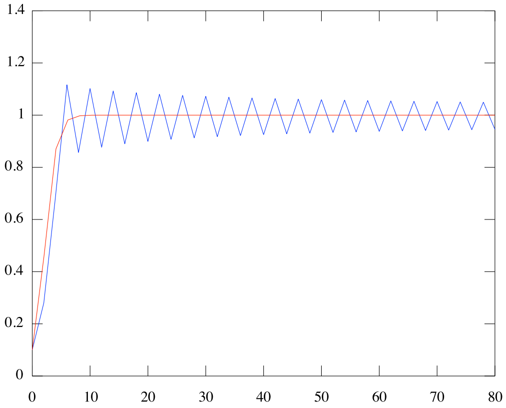


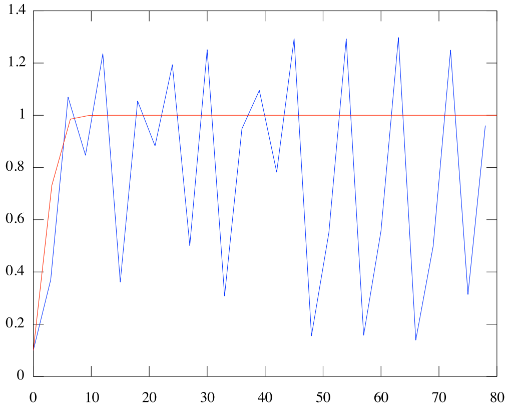
- How does the Euler method do for $\Delta t = 1$?



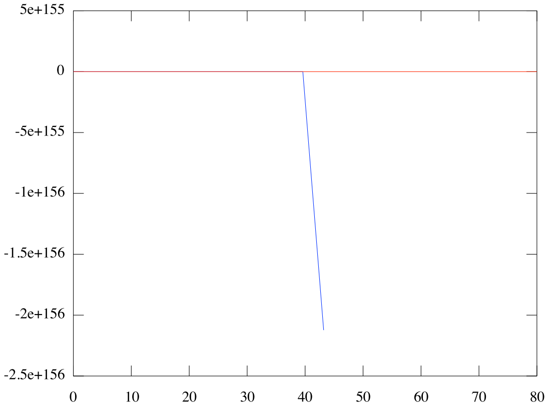
- How does the Euler method do for $\Delta t = 2$?



- How does the Euler method do for $\Delta t = 3$?



- How does the Euler method do for $\Delta t = 3.6$?



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2. In practice we use very small time steps for accuracy.

But, it shows that even though the solution of the model may be smooth, the solution of the numerics need not be. The numerics can even be chaotic!

Since we want the numerics to accurately reflect the dynamics, so that we can use the numerics to understand the system being modeled, we must keep a careful watch on the various sources of error.

- In many modeling situations, ensuring that

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- The classical error theory for numerical methods formulates this as a *convergence* question.

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- For example, for Runge-Kutta methods

$$\|T(U;\Delta t)\| = u(\Delta t) - U_1.$$

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- Roughly speaking, a numerical method has *order* r if for all sufficiently smooth functions $f(u)$ and all initial values U

$$\|T(U;\Delta t)\| = O(\Delta t^{r+1}) \text{ as } \Delta t \to 0.$$

- Defining the *global error* e_n at time $t = n\Delta t$ to be

$$e_n = \|u(n\Delta t) - U_n\|$$

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- there is a $\Delta t_c > 0$ such that for any $\Delta t \in (0, \Delta t_c)$ and $n\Delta t \in [0, T]$ the global error satisfies

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- But, we see that the bound grows exponentially in time and goes to infinity as $T \to \infty$. So what do we do if we are interested in long-time behaviour?

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- Much of the time we are interested in understanding the stability properties of the invariant sets (equilibrium points, periodic solutions, chaotic attractors, *etc.*) of our model dynamical system.
- Also, often our model contains parameters, which we may assume are constant, but may actually vary over longer time periods, or may just be subject to measurement or modeling error, so really we are dealing with a system

$$\dot{u}(t) = f(u, \mu), \quad u(0) = u_0, \quad n \in \mathbb{R}^+, \mathbb{R}, u \in \mathbb{R}^n, \mu \in \mathbb{R}^m \tag{2}$$

so we need to worry about bifurcations.

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- How do we know whether the invariant sets and bifurcations of the discretized system reflect or correspond to the invariant sets of our original model?

- Consider a simple linear example:

$$\dot{u} = Au, \quad u(0) = U, A = \begin{pmatrix} -\mu & 0 \\ 0 & -\mu/2 \end{pmatrix}, \mu > 1.$$

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  Clearly $U = 0$ is an equilibrium point, but it is only stable provided that the modulus of the eigenvalues of $I + \Delta t A$ are less than 1, *i.e.* if $|1 - \mu \Delta t| < 1$.

So we will run into trouble when $\mu\Delta t = 2$, *i.e.* when $\Delta t = 2/\mu$.

- Suppose that $\mu = 10$ and we let $U = (1,0)^T$. Then what happens for $\Delta t = 0.1/\mu$?

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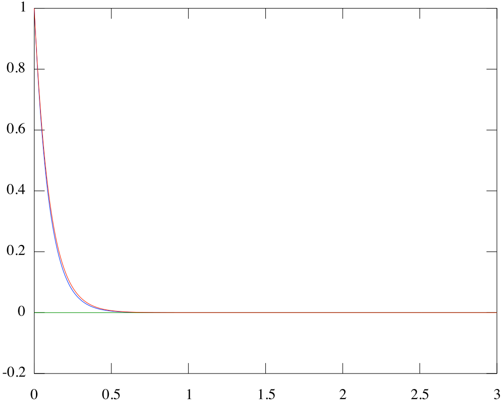

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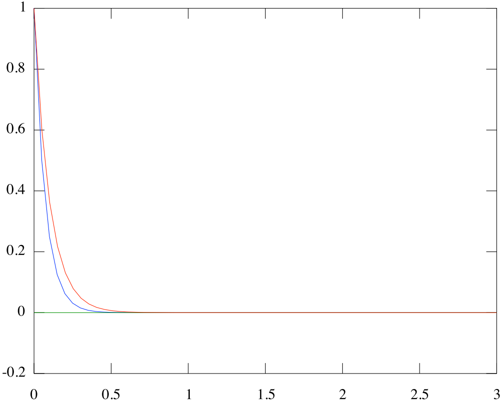

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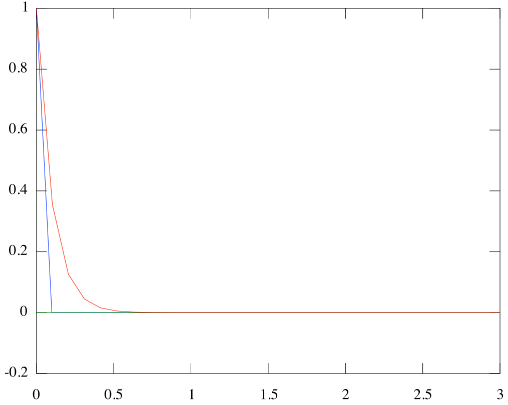

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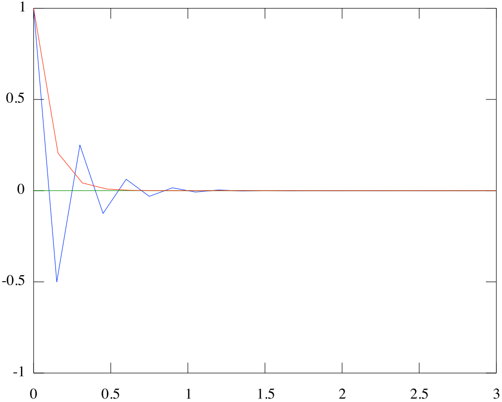

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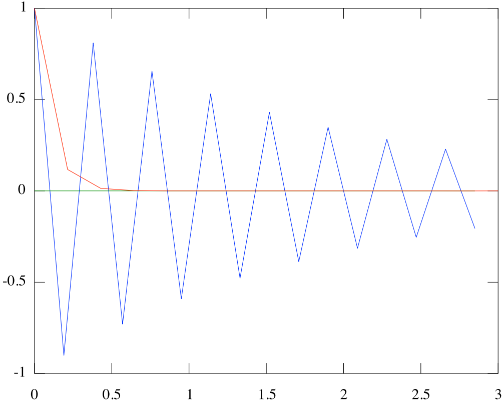

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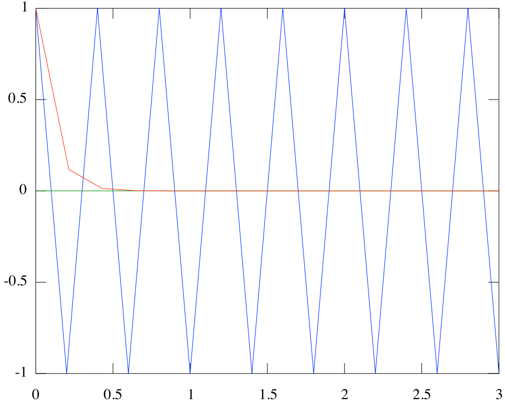

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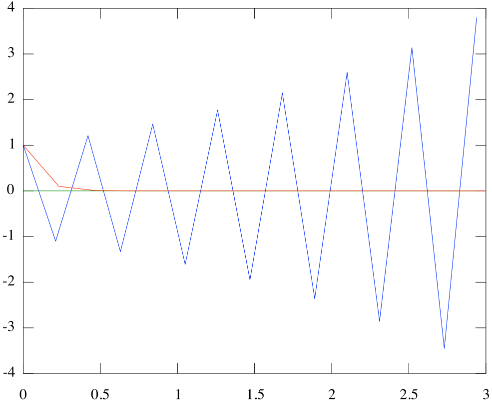

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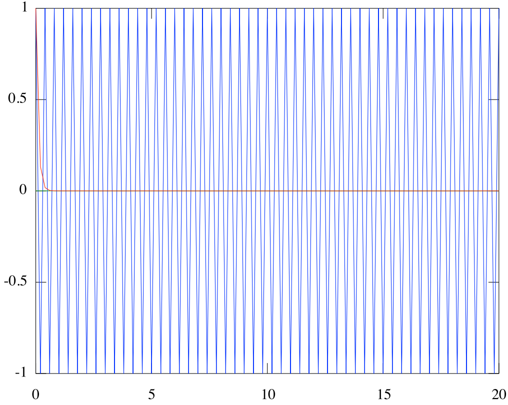

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How do we address these?

One approach is to treat the numerical methods as discrete dynamical systems so that the theory of dynamical systems can be brought to bear on the problem.

To get a taste for what this involves we need a few more definitions. . .

A (finite-dimensional) dynamical system can be characterized as a map from vectors to vectors.

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The maps (S^n and $S(t)$) are called *evolution semigroups*.

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We may see that equilibrium points and periodic solutions are invariant sets since they map onto themselves at time evolves.

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- Thus, the kind of questions we want to ask is do the ω-limit sets of our numerical method correspond to the ω-limit sets of the dynamical system.

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- When it does, then it is possible to prove results concerning the relationship between the invariant sets of $S(t)$ and $S^n_{\Delta t}$.

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- conditions under which structural properties, *e.g.* dissipativity or conservativeness, of the vector field f are preserved.
- conditions for convergence of the invariant sets of $S(t)$ and $S^n_{\Delta t}$, *e.g.* equilibriua, periodic solutions, invariant manifolds, *etc.*

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- the preservation of dynamical invariants, *e.g.* a Hamilitonian, phase volume, *etc.*, under discretization. This ensures stable numerics for computing integrable systems.

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- any fixed point of $S(t)$ is a fixed point of $S^n_{\Delta t}$; but
- specific conditions for critical values of Δt where spurious fixed points bifurcate from equilibria.

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- it is important to be aware of the kinds of distortion that discretization can produce when using numerics to study dynamical systems;
- treating numerics as discrete dynamical systems enables one to determine how to design numerical methods that stably compute continuous dynamical systems of various kinds.

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- Discretization of a model of a physical system.

Thank You!

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